Introduction to auctions

Vijay Kamble
Topics covered

1. Second price auction
2. First price auction
3. Mechanism design basics
4. Vickrey-Clarke-Groves mechanisms
5. Multi-item auctions
6. Spectrum Auctions
7. Sponsored search auctions and GSP
Introduction to auctions

• Auctions are mechanisms by which goods or services are allotted to multiple contenders by eliciting their willingness to pay
• Most common formats for a single good: First price auction, second price auction, dutch auction, english auction
• Several objectives: social optimality (efficiency), revenue optimality, budget balance
• More general auctions for multiple items: Vickrey-Clarke-Groves (VCG), Ausubel ascending price etc.
The basic set-up

• One seller with a single object to sell
• N potential buyers, i=1,…,N
• Each buyer i has a private value for the object $X_i$
• $X_i \sim F_i(x)$ with support on $[0, w_i]$, independent
• Quasi-linear utilities. If a buyer i gets the object and pays $p_i$ his net utility is $X_i - p_i$
The second price auction

• The seller elicits a bid $b_i$ from each buyer $i$, which is a report of maximum amount $i$ would pay for the object.
• The object is allotted to the buyer with the highest bid at the price of the second highest bid.
• Utility of buyer $i$ is

$$U_i = \begin{cases} 
  x_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\
  0 & \text{if } b_i < \max_{j \neq i} b_j 
\end{cases}$$
Equilibrium bidding

• Given the rules, the buyers play a game where the strategy is choosing the bid $b_i = \beta_i(x_i)$ as a function of $x_i$

• **Theorem:** In a second price auction, it is a dominant strategy for each buyer to bid truthfully, i.e. bid according to $\beta_i(x_i) = x_i$.
  
  – If you underbid when you are winning, either it doesn’t matter or you lose the auction
  
  – If you overbid when you are losing then either it doesn’t matter or you win the auction but pay more than your value.
Second price auction: properties

- The second price auction results in an *efficient* allocation: the object goes to the buyer who values it the most.
- It is *truthful*.
- It is *individually rational*: no winning buyer pays more than his valuation.
- It is not *budget-balanced*: sum of payments of the buyers is not zero. Why is this important?
  - Payments are being made even though the seller has no value for the object.
  - The efficient allocation requires payments purely because of lack of information.
English auction

• The second price auction is ‘weakly’ equivalent to the ascending price English auction.
  – The seller slowly increases the price and at any time a number of buyers are interested in the object at that price.
  – The seller continues to increase the price till only one buyer remains and he gets the object at the corresponding price.
• Thus the winning buyer gets the object at the price at which the second buyer drops out.
• The optimal policy is the same as that of the second price auction: stay till the price reaches your value and then drop out.
First price auction

• The seller elicits a bid $b_i$ from each buyer $i$, which is a report of maximum amount $i$ would pay for the object.

• The object goes to the highest bidder and he pays the amount he has bid.

$$U_i = \begin{cases} x_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

• Clearly you should underbid. How much?
Example

• Consider 2 buyers with valuations distributed uniformly in [0,1].
• Let a and b be their realized values.
• We can argue that their equilibrium bidding strategy should be \( \beta(x) = x/2 \)
  
  – What is the best response to this strategy?
  – Solve

\[
\text{arg max}_s (x - s)F(2s) \text{ for } s < 1/2
= \text{arg max}_s (x - s)2s
= x/2
\]
Equilibrium bidding

• Assume that $X_i \sim F_i = F$ for all $i$ and they are independent.

• We look for a symmetric equilibrium bidding strategy

  $\beta(x)$ increasing in $x$

  

  $\beta(x) = \arg \max_s (x - s) G(\beta^{-1}(s))$

  where $G$ is the c.d.f. of the highest of $n-1$ valuations of the other buyers, denoted by $Y$.

• Solving the first order condition, we get

  $\beta(x) = \frac{\int_0^x xg(x)}{G(x)}$

  

  $= E[Y \mid Y < x]$
Properties

• Efficient if all buyers symmetric.
• Strategically equivalent to a dutch decending price auction.
  – The seller decreases price from a high number slowly while eliciting desire to buy.
  – This continues till someone is willing to buy. That buyer is allotted the object at the corresponding price.

• What about revenue to the seller? How does it compare to the second price auction?
Revenue comparison

- Consider again the example of two buyers whose valuation is U[0,1].
- The revenue of the second price auction is $E(\min\{X_1, X_2\})$
- Revenue of the first price auction is $E(\max\{X_1/2, X_2/2\})$
Revenue comparison

- Consider again the example of two buyers whose valuation is $U[0,1]$.
- The revenue of the second price auction is $E(\min\{X_1, X_2\})$
- Revenue of the first price auction is $E(\max\{X_1/2, X_2/2\})$
- Both are the same. Is this a coincidence?
The revenue equivalence principle

• An auction is efficient if the object always goes to the one who values it the most at equilibrium.

• The revenue equivalence principle: Any efficient auction such that the expected payment of the buyer with valuation 0 is 0, yields the same expected revenue to the seller.

• For symmetric buyers, both the first and second price auctions are efficient. And expected payment at value 0 is 0. Hence they give the same revenue.
Proof

• **Step 1: Revelation principle**: Any auction can be implemented as an equivalent auction where truthful bidding is an equilibrium strategy and which gives the same revenue to the seller.

• **Step 2**: Consider equivalent truthful implementations of two efficient auctions. The for truthful auction 1, expected utility of a buyer $i$ whose value is $x$ but reports $y$ is

\[
U^1(x, y) = x\pi_i(y) - p_i^1(y)
\]

But since the auction is truthful, the optimal $y=x$. First order condition gives

\[
0 = x\pi_i'(x) - p_i^1'(x)
\]

\[
p_i^1(x) = p_i^2(x) = p_i(0) + \int_0^x y\pi_i'(y)\,dy
\]
Social choice functions

- A social choice function is a mapping from the preferences of agents to a set of alternatives.
- Suppose there are K alternatives, denoted by set A.
- There are N agents, whose preferences over the alternatives is captured by their ‘type’, which is a vector $v_i = (v_i^1, \cdots, v_i^K)$ which are their values for the K alternatives.
- The set of possible types for agent $i$ is $T_i \subseteq \mathbb{R}^K$.
- A social choice function is a mapping $f : T_1 \times \cdots \times T_n \to A$. 
Mechanisms

• Mechanisms are procedures to elicit the types of the agents so that a social choice function can be implemented.
• A mechanism elicits types from the agents and depending on the reports, selects an outcome and payments
• Let $L_1, \cdots, L_n$ be the sets of messages of the players

• A mechanism consists of:
  – An outcome function: $Q : L_1 \times \cdots \times L_n \rightarrow A$
  – A payment function: $M : L_1 \times \cdots \times L_n \rightarrow \mathbb{R}^n$

• Various notions of equilibrium reporting strategies
  – Dominant strategy
  – Bayes-Nash
  – Other....
Kinds of equilibria

- **Dominant strategy equilibrium**: \( \{\beta_i(v_i)\} \) is a dominant strategy equilibrium if

\[
v_i^Q(\beta_i(v_i), u^{-i}) - M_i(\beta_i(v_i), u^{-i}) \geq v_i^Q(t_i, u^{-i}) - M_i(t_i, u^{-i}) \quad \text{for all } i, u^{-i}, t_i
\]

- **Bayes-Nash equilibrium** assumes probability distribution on the type space for each agent \( i \). The equilibrium condition is

\[
E(v_i^Q(\beta_i(v_i), \beta^{-i}(V^{-i}))) - E(M_i(\beta_i(v_i), \beta^{-i}(V^{-i}))) \\
\geq E(v_i^Q(t_i, \beta^{-i}(V^{-i}))) - E(M_i(t_i, \beta^{-i}(V^{-i}))) \quad \text{for all } i, t_i
\]

where the expectation is over the randomness in \( V^{-i} \).
Implementability

• Suppose the alpha-equilibrium reporting strategies of the agents are given by \( \beta_i(v_i) : L_i \rightarrow T_i \)

• A mechanism implements a social choice function \( f \) under alpha-equilibrium if

\[
Q(\beta_1(v_1), \beta_2(v_2), \cdots, \beta_n(v_n)) = f(v_1, \cdots, v_n)
\]

for all \( v_1, \cdots, v_n \in T_1 \times \cdots \times T_n \)
Example: Allotting a single object

• For example consider the allocation of a single object
  – Set of alternatives is the choice of who to allot the object to
  – SCF: efficiency, or allot to the agent who values the most
  – Second price auction is a mechanism and it implements the SCF under dominant strategy equilibrium
  – First price auction implements the SCF under the bayes-nash equilibrium if the buyers are symmetric
Revelation principle

• A direct mechanism is one in which $L_i = T_i$ for all $i$.
• Revelation principle: If there is a mechanism that implements a social choice function under alpha-equilibrium, then there is a direct mechanism which implements the function such that truthful reporting is an alpha-equilibrium.
• Thus implementability reduces to the question ‘is there a payment rule which implements a SCF truthfully?’
Vickrey Clarke Groves mechanism

- There is a generic mechanism which implements the social welfare maximizing social choice function $f$ in a dominant strategy equilibrium.

$$f(v_1, \cdots , v_n) = \arg \max_{a \in A} \sum_{i=1}^{n} v^a_i$$

- It is called the VCG mechanism
- The allocation rule is

$$Q(b_1, \cdots , b_n) = a^* = \arg \max_{a \in A} \sum_{i=1}^{n} b^a_i$$

- Payment rule is

$$M^i(b_1, \cdots , b_n) = \max_{a \in A} \sum_{j=1; j \neq i}^{n} b^a_j - \sum_{j=1; j \neq i}^{n} b^a_{j}$$
Equilibrium reporting strategy

• Suppose buyer $i$ has type $v_i$ but he reports $w_i$. Then for any report from the other buyers, his utility is

$$U(w_i, b^{-i}) = v_i^Q(w_i, b^{-i}) + \sum_{j=1; j \neq i}^n b_j^Q(w_i, b^{-i}) - \arg \max_{a \in A} \sum_{j=1; j \neq i}^n b_j^a$$

This part is maximized by choosing $w_i = v_i$

• Thus truthful reporting is a dominant strategy equilibrium

• Second price auction is a special case of this mechanism
Example: Multi-unit auctions

- Suppose a seller has M identical items for sale.
- N agents and each agent i has a utility $U_i(x_i)$ for $x_i$ items.
- $U_i$ are ‘concave’ i.e. $U_i(x + 1) - U_i(x)$ is decreasing in $x$.
- We want to implement the efficient allocation

$$\max \sum_{i=1}^{N} U_i(x_i)$$

s.t. $\sum_{i=1}^{N} x_i \leq M.$
Example

• Three buyers a, b and c and 4 items.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta U_a(x) = u_a(x)$</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u_b(x)$</td>
<td>8</td>
<td>5</td>
<td>2.2</td>
<td>0</td>
</tr>
<tr>
<td>$u_c(x)$</td>
<td>4.5</td>
<td>4</td>
<td>2.5</td>
<td>0</td>
</tr>
</tbody>
</table>

• 4 is the minimum price that clears the market
Example

• Three buyers a, b and c and 4 items.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta U_a(x) = u_a(x)$</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u_b(x)$</td>
<td>8</td>
<td>5</td>
<td>2.2</td>
<td>0</td>
</tr>
<tr>
<td>$u_c(x)$</td>
<td>4.5</td>
<td>4</td>
<td>2.5</td>
<td>0</td>
</tr>
</tbody>
</table>

• 4 is the minimum price that clears the market
• If I say that I will choose the minimum price that clears the market, will the buyers reveal their demand functions?
Example

• Three buyers a, b and c and 4 items.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$\Delta U_a(x) = u_a(x)$</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u_b(x)$</td>
<td>8</td>
<td>5</td>
<td>2.2</td>
<td>0</td>
</tr>
<tr>
<td>$u_c(x)$</td>
<td>4.5</td>
<td>4</td>
<td>2.5</td>
<td>0</td>
</tr>
</tbody>
</table>

• 4 is the minimum price that clears the market
• If I say that I will choose the minimum price that clears the market, will the buyers reveal their demand functions?
• There is an incentive for ‘demand reduction’. ‘I would rather have lower number of items for a lower price per item than have the optimal number for a high price per item’
• We can use VCG but in that case the buyers have to convey their entire utility functions.
• Further you have to compute the optimal allocations for N+1 problems
• Is there a communication efficient way to implement VCG just like the english auction implemented the second price auction?
• Ausubel describes such an ascending price auction.
Ausubel ascending price auction

• The seller slowly raises the prices from 0 according to some function $p(t)$
• The users submit demands at these prices at each time.
• At any time let $D_i(p(t))$ be the number of items demanded by buyers other than i.
• $C_i(t) = [M - D_i(p(t))]^+$ are the non-demanded items which is an increasing function.
• Every time the non-demanded items for i increases, he gets an additional item at the corresponding price.
• The auction ends when the total demand of the agents equals supply.
Example

• Three buyers a, b and c and 4 items.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$\Delta U_a(x) = u_a(x)$</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u_b(x)$</td>
<td>8</td>
<td>5</td>
<td>2.2</td>
<td>0</td>
</tr>
<tr>
<td>$u_c(x)$</td>
<td>4.5</td>
<td>4</td>
<td>2.5</td>
<td>0</td>
</tr>
</tbody>
</table>

• 1: (2,3,3)
• 2: (1,3,3)
• 2.2: (1,2,3) --- c gets allotted 1 object at price 2.2
• 2.5: (1,2,2) --- b gets allotted 1 object at price 2.5
• 4: (1,2,1) --- b and a get allotted 1 object at price 4. Demand matches supply and the auction ends.
Spectrum auctions

- Spectrum is a national resource that the government sells to private entities to make revenue.
- An auction mechanism used for the first time in 1994 first to sell 10 paging licenses and later to sell 99 broadband licenses. Generated $617 million and 7 billion resp.
First approach: efficiency via VCG

- M licenses are to be sold to N buyers at once.
- The buyers have a value function defined on each of the $2^N$ possible subsets of these licenses. A bid would be this value function!
- Once this function is communicated, the auctioneer has to solve $N+1$ combinatorial optimization problems to implement an efficient allocation.
- We instead want a mechanism that reduces the communication overhead, which is easy to implement and (hopefully) preserves some nice properties.
The increasing prices approach

- How about using an allocation rule inspired from the english auction or the ausubel auction?
- Slowly increase the prices for the licenses at fixed increments (according to a specified rule).
- At each time each buyer submits the set of licenses that it is interested in buying at those prices, determined by solving the utility maximization problem.
- The auction ends at the price vector where there are no conflicts in the demand and each license is sold: each license is demanded by exactly one buyer. This situation is called ‘Competitive equilibrium’.
Does it work?

• Does a competitive equilibrium exist?
• Does the process converge to this equilibrium?
• Yes for both, if
  – the licenses are mutual substitutes: raising prices for a set of licenses does not reduce the demand for licenses not in that set.
  – Buyers are price taking (they do not anticipate the effect on the prices by their bids).

Refer to:
‘Putting auction theory to work’ Paul Milgrom
‘An efficient dynamic auction for heterogeneous commodities’ Lawrence Ausubel
For a tutorial on competitive equilibrium, ‘General equilibrium’ Jonathan Levin
The actual implementation : SMRA

• Simultaneous multiple round ascending auction: bidding takes place in rounds.
• At each round bidders make sealed bids for all the licenses that they are interested in. The new bids, bidders and the ‘standing high bid’ is posted.
• In any round the minimum bid increment is larger of some fixed amount or a fixed percentage of the standing high bid of previous round.
• Bids may be withdrawn but the buyer pays the difference as a penalty if the final price is lower than the bid.
Activity rules and ending rules

- A buyer buys ‘eligibility’ to bid on a quantity of spectrum by paying a deposit at the beginning of the auction.
- At each round he has to be ‘active’ on at least a fixed fraction $f$ of his eligible quantity. Active implies he either owns the previous standing high bid or he makes a new valid bid.
- If his eligibility is $x$ and he is active on quantity $y$ less than $f \cdot x$ then his eligibility drops to $y/f$.
- The auction ends when there is no new bid on ANY ONE of the licenses (!!)
Sponsored search auctions

Almost entirely based on

What happens when you search for dental insurance on Google?
Position auctions: auctions of keywords

User searched for ‘Dental insurance’

Advertisers have a value per click from a user who searched for ‘dental insurance’. Place a bid for amount willing to pay per click.

Google has different positions to place ad, different Click-through-rates (probability of getting a click)
Some unique characteristics

• Bids can be continuously changed, every time a keyword is searched, the current bid is applied.

• Pay-per-click model: Host would like advertisers to pay for every display, advertisers would like to pay only for a successful transaction. Pay-per-click is a convenient middle-ground.

• Single bid for each keyword, even though multiple positions for sale.

• Implicitly assumes that the value for a click does not depend on which position it was placed. The position only affects the probability of getting a click.
Common setting

- Advertisers $i=1,...,K$ and positions $j=1,...,N$. Positions are allotted to advertisers in decreasing order of bids.
- $s_i$, $b_i$ is the value per click and bid resp. of advertiser $i$.
- $b_{(j)}$, $g(j)$ is the bid and the identity of the advertiser in position $j$.
- $\alpha_j$ is the CTR, number of clicks in a fixed duration in position $j$.
- Complete information. Everyone knows everyone else’s valuations

$\alpha_1$

$\alpha_2$

$\alpha_3$
Some history: Generalized first price auction (Yahoo!/Overture)

- Advertisers are allotted in decreasing order of their bids.
- Each advertiser pays his bid per click.
- Example: 2 slots, 3 advertisers $i=1,2,3$ with value per click $\$10$, $\$4$ and $\$2$ resp.

1. $\alpha_1 = 200$
   - Advertiser 2 will not want to bid more than 2.01 for the second slot.

2. $\alpha_2 = 100$
   - Advertiser 1 will not want to bid more than 2.02 for the first slot.
   - But then advertiser 2 will want the first slot for 2.03 and so on..
   - There is no pure strategy equilibrium! Bidding is unstable.
The grand fix by Google: Generalized second price auction

- Advertisers are allotted in decreasing order of their bids.
- Each advertiser pays the bid of the next highest advertiser per click.
- Example: 2 slots, 3 advertisers i=1,2,3 with value per click $10, $4 and $2 resp.

1. $\alpha_1 = 200$

2. $\alpha_2 = 100$

- Truth-telling is an equilibrium.
- Advertiser 1 pays $4$ p.c. for 200 clicks in slot 1 and 2 pays $2$ p.c. for 100 clicks in slot 2.
- Changing bids does not benefit anyone.
What inspired the fix?

• Marketing materials say that Google’s “unique auction model uses Nobel Prize-winning economic theory to eliminate... that feeling that you’ve paid too much”

• “Roth and Ockenfels (2002) describe another example in which the architects...may have tried to implement a mechanism strategically equivalent to the Vickrey auction, but did not get an important part of the mechanism right.” Edelman et.al (2007)

• So it seems they were trying to implement VCG. Clearly GSP does it if there is only one position.
What is wrong?

• Let us look at our example again and modify it a bit.

• 2 slots, 3 advertisers i=1,2,3 with value per click $10, $4 and $2 resp.

\[ \alpha_1 = 200 \]

\[ \alpha_2 = 199 \]

• If all players bid truthfully, then 1’s payoff is \( 200 \times (10-4) = $1200 \)

• But if 1 slightly underbids 2, then he gets slot 2 for price per click 2 and gets a payoff \( 199 \times (10-2) = $1592 > $1200 \)

• GSP is not truthful!
The mechanism they got wrong: VCG

- VCG ranks and places advertisers in decreasing order of their bids.
- Each player pays the externality he imposes on others.
- Payment of the last advertiser who gets a slot is 0 if $N \geq K$ and $\alpha_N b^{N+1}$ otherwise.
- For all $i < \min(N,K)$, $p^V, (i) = (\alpha_i - \alpha_{i+1}) b^{(i+1)} + p^V, (i+1)$
A comparison

- Example: 2 slots, 3 advertisers $i=1,2,3$ with value per click $\$10$, $\$4$ and $\$2$ resp.

  \[ \alpha_1 = 200 \]

  \[ \alpha_2 = 100 \]

  **GSP**

  - Truth-telling is an equilibrium.
  - Advertiser 1 pays $\$4$ p.c. for 200 clicks in slot 1 and 2 pays $\$2$ p.c. for 100 clicks in slot 2.
  - Total revenue: $800 + 200 = \$1000$
A comparison

• Example: 2 slots, 3 advertisers $i=1,2,3$ with value per click $\$10$, $\$4$ and $\$2$ resp.

  \[ \alpha_1 = 200 \]

  \[ \alpha_2 = 100 \]

GSP

• Truth-telling is an equilibrium.
• Advertiser 1 pays $\$4$ p.c. for 200 clicks in slot 1 and 2 pays $\$2$ p.c. for 100 clicks in slot 2.
• Total revenue: $800 + 200 = \$1000$

VCG

• Truth-telling is an equilibrium.
• Advertiser 1 pays $\$100 \times 4 + 100 \times 2$ TOTAL for 200 clicks in slot 1 and 2 pays $\$100 \times 2$ TOTAL for 100 clicks in slot 2.
• Total revenue: $600 + 200 = \$800 < \$1000$!!
A comparison

- Example: 2 slots, 3 advertisers i=1,2,3 with value per click $10, $4 and $2 resp.

  \[ \alpha_1 = 200 \]
  \[ \alpha_2 = 100 \]

GSP
- Truth-telling is an equilibrium.
- Advertiser 1 pays $4 p.c. for 200 clicks in slot 1 and 2 pays $2 p.c. for 100 clicks in slot 2.
- Total revenue: 800 + 200=$1000

VCG
- Truth-telling is an equilibrium.
- Advertiser 1 pays $100*4 +100*2 TOTAL for 200 clicks in slot 1 and 2 pays $100*2 TOTAL for 100 clicks in slot 2.
- Total revenue: 600+200=$800< $1000 !

REMARK 1: If the bids are the same under the two mechanisms, the revenue of GSP is at least as high as VCG. (😊)
Equilibrium analysis

- Consider a related matching game: N objects, K buyers.
- Value of matching object buyer i with player j is $s_i \alpha_j$
- (Shapley and Shubik 1971) Then there exist a set of stable assignment price vectors for the objects that
  1. Demand balances supply: there is a stable matching
  2. The resulting allocation is efficient
  3. There exists a price vector which is seller optimal: it dominates every other price vector
  4. These is a price vector which is buyer optimal: it is dominated by every other price vector
  5. (Demange, Gale, Sotomayor 1981) The buyer optimal price vector is precisely the VCG price vector.
Correspondence between stable assignments and a class of equilibria of GSP

• There is a nice one-to-one correspondence between a class of equilibria of GSP with the set of stable assignment price vectors when the number of advertisers is at least as many as the number of ad slots

• Locally envy free equilibria: An equilibrium of GSP is locally envy-free if a player cannot improve his payoff by exchanging bids with the player ranked one position above him.

• This is sufficient to ensure that a player cannot improve his payoff by exchanging bids with any player ranked above him.

• Equilibrium condition ensures that he cannot improve by exchanging bids with any player below him (because otherwise he can just underbid that player to pay the price that he pays)
Mapping

• Now the correspondence is simple.
• Locally envy free equilibria ➔ A stable matching price vector is trivial by the definition.
• A stable matching price vector ➔ Locally envy free equilibria

1. Say $p_j$ is the stable assignment price of slot $j$
2. Then highest value player bids truthfully and $j+1$ st highest value player bids $p_j/\alpha_j$
3. Verify that this leads to bids being increasing
4. Verify that this is an equilibrium and it is locally envy free
Enter VCG

• As mentioned before, there is a ‘smallest’ stable assignment price vector which corresponds to the VCG prices.

• This vector corresponds to a locally envy-free equilibrium by construction. This equilibrium thus gives the VCG revenue, which is the smallest revenue to the advertiser amongst all LEF equilibria. 😊

• (Sticking to GSP doesn’t seem so dumb after all. Indeed a mistake well made.)

Further references:
“Position auctions” Hal Varian
Chapter “Sponsored search auctions” in the AGT book.
References

- AGT, Nisan et. al.
- SNR, Walrand and Parekh
- Auction theory, Vijay Krishna
- ‘An efficient ascending bid auction for multiple objects’ L. Ausubel, American economic review, 1997
- ‘Putting auction theory to work’ Paul Milgrom
- An efficient dynamic auction for heterogeneous commodities’ Lawrence Ausubel