Routing games

Vijay Kamble

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Example: one unit of traffic (e.g., cars) wants to go from $s$ to $t$

delay = 1 hour
(no congestion effects)

$\overset{\text{long + wide}}{s} \quad \overset{\text{short + narrow}}{t}$

delay in hours = fraction of traffic on edge
.depends on congestion

Question: what will selfish network users do?
A multi-commodity flow network is described by a directed graph $G = (V, E)$ and a set of source destination vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$. Parallel edges allowed. A vertex can participate in multiple source-sink pairs.

Let $P_i$ be the set of $s_i - t_i$ paths and $P = \bigcup_{i=1}^{k} P_i$.

A flow $f$ is a non-negative vector indexed by $P$. $r$ is a non-negative vector of traffic rates indexed by the set of source-sink pairs.

$f_P$ for $P \in P_i$ is the flow of the source sink pair $i$. $f_e = \sum_{P \in P: e \in P} f_P$ is the total amount of flow on edge $e$.

Flow $f$ is feasible for $r$ if for all $i$, $\sum_{P \in P_i} f_P = r_i$.

Cost function $c_e(.)$ denotes the congestion cost faced by traffic on edge $e$ as a function of $f_e$. 
Wardrop equilibrium: A simple behavioral principle

- For each source-sink pair \( i \), the delays on all the paths actually used are equal, and less than those which would be experienced by a single packet on any unused path.
- Let \( c_P(f) \) be the congestion cost on path \( P \in \mathcal{P}_i \).
  \[
  c_P(f) = \sum_{e \in P} c_e(f_e).
  \]

**Definition**

A feasible flow \( f \) is a Wardrop equilibrium if, for every source-sink pair \( i \), and every \( P, \tilde{P} \in \mathcal{P}_i \) with \( f_P > 0 \),

\[
  c_P(f) \leq c_{\tilde{P}}(f).
\]

- Related to Nash Equilibrium: No packet has an incentive to change its route.
Does it exist?

- Always exists and is cost unique.

**Proposition**

Let \((G, r, c)\) be an instance of a selfish routing network. Then

- There is at least one wardrop equilibrium.
- It is cost unique, i.e. if \(f\) and \(\bar{f}\) are two wardrop equilibria, then
  \[
  c_e(f_e) = c_e(\bar{f}_e) \quad \text{for each edge } e \in E.
  \]
Proof

Express the condition as the following variational inequalities for all $i = 1 \cdots, k$.

$$(c_P(f) - \lambda_i)f_P = 0$$

$$c_P(f) - \lambda_i \geq 0 \text{ for all } P \in P_i$$

$$f_P \geq 0$$

$$\sum_{P \in P_i} f_P = r_i$$

Expanding

$$(\sum_{e \in P} c_e(f_e) - \lambda_i)f_P = 0$$

$$\sum_{e \in P} c_e(f_e) - \lambda_i \geq 0 \text{ for all } P \in P_i$$

$$f_P \geq 0$$

$$\sum_{P \in P_i} f_P = r_i$$
Proof

These conditions can be seen as the first order optimality conditions of the optimization problem

$$\min_f \sum_{e \in E} \int_0^{f_e} c_e(x) dx$$

s.t. $\sum_{P \in \mathcal{P}_i} f_P = r_i \forall i$

$$f_P \geq 0 \forall P \in \mathcal{P}_i, \forall i.$$ 

The objective function is convex and continuous since it is an integral of an increasing continuous function. The set of feasible flows is compact. Thus the optimal value is unique.

This is called a *potential function* of a game. The equilibrium of a game can be found by solving a global optimization problem of minimizing the potential (Monderer and Shapley (96)).
Proof

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\[
\min_{f} \sum_{e \in E} \int_{0}^{f_e} c_e(x) \, dx
\]

\[
\text{s.t. } \sum_{P \in \mathcal{P}_i} f_P = r_i \ \forall i
\]

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f_P \geq 0 \ \forall P \in \mathcal{P}_i, \ \forall i.
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- The objective function is convex and continuous since it is an integral of an increasing continuous function. The set of feasible flows is compact. Thus the optimal value is unique.

- This is called a potential function of a game. The equilibrium of a game can be found by solving a global optimization problem of minimizing the potential (Monderer and Shapley (96)).
Price of anarchy

- Social optimum maximizes $\sum_{e \in E} f_e c_e(f_e)$.
- Consider the example by Pigou (1920):

\[ \text{Cost of Nash} = 1 \]
\[ \text{Cost of OPT} = \frac{3}{4} \]

\[
\text{cost of Nash flow} = \frac{4}{3} \times \text{cost of opt flow}
\]

aka price of anarchy

[Panadimitriou 01]
But another example:

Bad Example: \((r = 1, \ d \ large)\)

Nash flow has cost 1, min cost \(\approx 0\)

\(\Rightarrow\) Nash flow can cost arbitrarily more than the optimal (min-cost) flow
Price of anarchy

- Price of anarchy is defined as the worst case ratio of cost under wardrop and the social optimum cost.

**Definition**

The price of anarchy $\rho(G, r, c)$ of an instance $(G, r, c)$ is

$$\rho(G, r, c) = \frac{C(f)}{C(f^*)}$$

where $f$ and $f^*$ are the wardrop equilibrium and the social optimum flow profiles resp. Price of anarchy of a non-empty set $\mathcal{I}$ of instances is

$$\sup_{(G, r, c) \in \mathcal{I}} \rho(G, r, c).$$
Linear cost functions

Theorem (Roughgarden, ’00)

The price of anarchy of any network with affine cost functions of the form
\[ C = \{ ax + b : a, b \geq 0 \} \] is \( 4/3 \).

- Lower bound is given by Pigou example. Turns out it is the worst case.
- Same for concave cost functions.
- In fact price of anarchy is in general always attained for simple two node networks which capture Pigou’s bound.
Proof

First define a generalized pigou bound.

Definition

Let $\mathcal{C}$ be a non-empty set of cost functions. Then the pigou bound $\alpha(\mathcal{C})$ of $\mathcal{C}$ is given by

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{0 \leq x \leq r, r \geq 0} \frac{r c(r)}{x c(x) + (r - x) c(r)} = \sup_{c \in \mathcal{C}} \sup_{x \geq 0, r \geq 0} \frac{r c(r)}{x c(x) + (r - x) c(r)}$$

Intuition: fix flow $r$ and cost function $c$. Then imagine a 2 node network with a link with constant cost $c(r)$ and a link with cost function $c(x)$. Then the wardrop equilibrium cost is the numerator. Denominator captures the benefit of splitting the flow.

For affine cost functions, this bound evaluates to $\frac{4}{3}$.
Proof

- From the pigou bound, for \( x, r \geq 0 \)
  \[
  xc(x) \geq \frac{rc(r)}{\alpha(C)} + (x - r)c(r)
  \]

- If \( f^* \) and \( f \) are the optimal and wardrop flows respectively, then applying the above inequality to each edge in an instance \((G, r, c)\), we have:
  \[
  C(f^*) = \sum_{e \in E} c_e(f_e^*)f_e^* \\
  \geq \frac{\sum_{e \in E} c_e(f_e)f_e}{\alpha(C)} + \sum_{e \in E} (f_e^* - f_e)c(f_e) \geq \frac{C(f)}{\alpha(C)}
  \]

- Second inequality because \( \sum_{e \in E} (f_e^* - f_e)c(f_e) \geq 0 \) for any wardrop equilibrium flow \( f \) and any other flow \( f^* \). Because from definition, flow is wardrop equilibrium iff
  \[
  \sum_{P \in P_i} c_P(f_P)f_P \leq \sum_{P \in P_i} c_P(f_P)f^*_P
  \]
Adding links to a network can increase cost at Wardrop equilibrium!

Cost of Nash flow = 1.5

Cost of Nash flow = 2