Notes for EE290Q: Optimal Auction

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Abstract
This note explains Myerson’s paper on optimal auctions.

1 Introduction
A single item is to be sold to one of \( N \) agents who have independent valuations whose distributions are known to the seller. Myerson derived the auction with the maximum expected revenue among all incentive compatible and individually rational auctions.

2 Terminology
We consider the auction of a single item to agents with private valuations. The agents report their bids to the auctioneer. The auctioneer then decides who gets the item and how much each agent pays according to rules that he made known to the agents and that the seller is committed to obeying.

We have the following definitions:

- The revenue of the auction is the amount that the agents pay to the seller.
- The surplus of an agent who gets the item is his valuation for the item minus what he pays for it. His surplus is zero if he does not get the item and does not pay.
- An auction is Individually Rational (IR) if the payment of every agent never exceeds his valuation.
- An auction is Incentive Compatible (IC) if the expected surplus of every agent is maximized when he bids his true valuation, assuming that the other agents bid their true valuation. In other words, truth-telling is a Bayes-Nash equilibrium.
- An auction is optimal if it generates the largest expected revenue among all IR and IC auctions.
3 The Result

There is a single item and $N$ users with independent valuations $V_1, \ldots, V_N$ for the item. Valuation $V_i$ takes values in $[a_i, b_i]$ and has pdf $f_i$ and cdf $F_i$ known to the seller.

We assume that

$$c_i(x) := x - \frac{1 - F_i(x)}{f_i(x)}$$

is non-decreasing in $x \in [a_i, b_i]$. The number $c_i(x)$ is called the virtual valuation of user $i$ if his valuation is $x$.

For instance, if $V_i$ is uniform in $[a_i, b_i]$, then

$$c_i(x) = x - \frac{(b_i - x)/(b_i - a_i)}{1/(b_i - a_i)} = 2x - b_i.$$

Theorem 1 (Myerson): Let $B_i$ be the bid of agent $i = 1, \ldots, N$. The following auction is IR, IC and optimal: Allocate the item to the agent with the largest value of $c_i(B_i)$ if it is positive. That agent then pays $p_i(B_1, \ldots, B_N)$ where

$$p_i(B_1, \ldots, B_N) = \min\{B \mid c_i(B) > c_j(B_j), \forall j \neq i \text{ and } c_i(B) \geq 0\}.$$

If all $c_i(B_i)$ are negative, do not allocate the item.

Thus, $p_i(B_1, \ldots, B_N)$ is the minimum bid that would make agent $i$ win the auction.

4 Examples

Uniform $[0, 1]$

Assume that the valuations $V_i$ are uniformly distributed in $[0, 1]$. Then

$$c_i(x) = x - \frac{1 - x}{1} = 2x - 1.$$  

Thus, the item is allocated to the user with the largest valuation $V_i$, provided that it is larger than 0.5. Also, the payment is the maximum of 0.5 and the second largest valuation.

Thus, this auction is a second price auction with a reserve price of 0.5. This auction can be implemented as an ascending auction that starts with the reserve price 0.5. Note that in this particular case, the item always goes to the agent with the largest valuation, so that it is efficient (with the exception that no one gets the item if all the valuations are less than 0.5).
**Uniform** \([a_i, b_i]\)

Assume that the valuations \(V_i\) are uniform in \([a_i, b_i]\). Then the item goes to the agent with the maximum value of \(2B_i - b_i\), provided that it is positive. Also, the payment is the smallest value of \(B\) such that

\[
2B - b_i \geq 2B_j - b_j, \forall j \neq i \text{ and } 2B - b_i \geq 0.
\]

The item might be allocated to an agent who does not value it the most. Thus, the auction is not necessarily efficient.

## 5 Proof of Theorem

### Notation

An auction is characterized by

\[
\pi_i(B) = P[ \text{agent } i \text{ gets item } | \text{ bids are } B]
\]

and

\[
p_i(B) = \text{price that agent } i \text{ pays when bids are } B.
\]

We denote by \(V_{-i}\) the vector of all the valuations other than that of agent \(i\).

Let

\[
\pi_i(u) = E(\pi_i(u, V_{-i})) \text{ and } p_i(u) = E(p_i(u, V_{-i}).
\]

Finally,

\[
S_i(x) = x\pi_i(x) - p_i(x)
\]

is the expected surplus of agent \(i\) when his valuation is \(x\).

### Proof Steps

The proof has the following steps:

(A) An auction is IC if and only if

\[
\pi_i(B_i) \text{ is nondecreasing}
\]

and

\[
p_i(u) = u\pi_i(u) - \int_{a_i}^{u} \pi_i(x)dx - S_i(a_i).
\]

(B) If an auction is IC, its expected revenue is

\[
E(\sum_i p_i(V)) = E(\sum_i \pi_i(V)c_i(V)) - \sum_i S_i(a_i).
\]

(C) The Myerson auction is IR, IC and maximizes the expected revenue over all IC and IR auctions.
Note that (A) is a result about Bayes-Nash equilibrium. That is, if one assumes that all the agents other than agent $i$ are bidding their true valuation, then one can calculate $p_i(u)$ and $\pi_i(u)$ and it is optimal for agent $i$ to bid his true valuation if and only if (1) and (2) hold. Thus, truth-telling is a Bayes-Nash equilibrium if and only if (1) and (2) hold.

In statement (B), one assumes that all the agents bid their true valuation, so that one can calculate the expected revenue of the auction. Note that this expected revenue depends only on the allocation rule $\pi_i(V)$ and on $S_i(a_i)$. This is the revenue equivalence theorem. For instance, all auctions that award the item to the agent who values it the most and do not award it to an agent $i$ whose valuation is minimum have the same expected valuation. Examples are the first and second price auctions in a symmetric situation (i.e., with i.i.d. valuations).

**Proof of (A), “Only if: $\Rightarrow$”**

If the auction is IC, it must be that

$$S_i(V_i) = \max_x \{ V_i \pi_i(x) - p_i(x) \}.$$

This implies that $S_i(V_i)$ is convex. (It is the maximum of linear functions.) Also,

$$V_i \pi_i(x) - p_i(x) = x \pi_i(x) - p_i(x) + [V_i - x] \pi_i(x) = S_i(x) + [V_i - x] \pi_i(x).$$

If the auction is IC, the maximum of this expression occurs at $x = V_i$, so that the derivative with respect to $x$ is equal to zero at $x = V_i$, so that

$$S_i'(V_i) = \pi_i(V_i).$$

Hence, $\pi_i(V_i)$ is nondecreasing, being the derivative of a convex function. Also,

$$S_i(u) = S_i(a_i) + \int_{a_i}^u \pi_i(x)dx.$$

Since $S_i(u) = u \pi_i(u) - p_i(u)$, we get (2).

**Proof of (A), “If: $\Leftarrow$”**

Let $U_i(x) = V_i \pi_i(x) - p_i(x)$ be the surplus of agent $i$ when he declares that his valuation is $x$. Then

$$U_i'(x) = V_i \pi_i'(x) - p_i'(x).$$

Now, (2) implies that $p_i'(x) = x \pi_i'(x)$. Hence,

$$U_i'(x) = (V_i - x) \pi_i'(x).$$

Since $\pi_i'(x) \geq 0$, it follows that $U_i(x)$ is maximized at $x = V_i$, so that the auction is IC.
Proof of (B)

From (1) we get

\[ E(p_i(V_i)) = E(V_i\pi_i(V_i)) - E(\int_{a_i}^{V_i}\pi_i(x)dx) - S_i(a_i). \]  

(4)

Now,

\[ E(\int_{a_i}^{V_i}\pi_i(x)dx) = \int_{a_i}^{b_i} \int_{\pi_i^{-1}(v)}^{b_i} f_i(v)dv, \]

by changing the order of integration

\[ = \int_{a_i}^{b_i} [1 - F_i(x)]\pi_i(x)dx = \int_{a_i}^{b_i} \frac{1 - F_i(x)}{f_i(x)}\pi_i(x)f_i(x)dx = E(1 - F_i(V_i))\pi_i(V_i), \]

which, together with (4) proves (3).

Proof of (C)

The auction is IR. That is, \( V_i \geq \pi_i(V) \). Indeed, if agent \( i \) gets the item, then \( c_i(V_i) \) is large enough to make him win the auction. Also, \( p_i(V_i) \) is the smallest value of \( B_i \) so that he wins the auction, so that it is necessarily smaller than or equal to \( V_i \).

The auction is IC. To see this, note that \( \pi_i(V_i) \) is nondecreasing since a larger \( V_i \) implies a larger \( c_i(V_i) \) and therefore a larger probability of winning the auction. Also, we claim that (2) is satisfied. To see this, note that, by definition of the auction,

\[ \pi_i(x, V_{-i}) = 1\{x \geq p_i(V)\}. \]

Consequently,

\[ \int_{a_i}^{u} \pi_i(x, V_{-i})dx = u\pi_i(u, V_{-i}) - p_i(u, V_{-i}). \]

This implies that (1) holds with \( S_i(a_i) = 0 \). Note also that the Myerson auction is such that \( S_i(a_i) = 0 \) because \( c_i(B) \geq 0 \) requires \( B \geq a_i \).

Finally, we show that the auction maximizes (3) over all IC and IR auctions. To see this, note that an IR auction is such that \( S_i(a_i) \geq 0 \). Also, the Myerson auction maximizes the term

\[ E(\sum_{i} \pi_i(V)c_i(V_i)) \]

since it allocates the item (i.e., selects \( \pi_i(V) = 1 \)) to the user \( i \) with the largest value of \( c_i(V_i) \) if it is positive.
References
